Theory of the Reflection of Light near the Polarising Angle.

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Supposing the phenomena of light to be due to displacements in a rotational ether, we have the following expressions for the kinetic energy and work function in a transparent isotropic medium:—

$$\mathrm{T} = \tfrac{1}{2}\int \rho\,(\dot\xi^2+\dot\eta^2+\dot\zeta^2)\,d\mathrm{V},$$

$$\mathrm{W} = \tfrac{1}{2}c^2\int \rho\mu^2(f^2+g^2+h^2)\,d\mathrm{V}.$$

Here $d\mathrm{V}$ denotes an element of volume, ρ is the density, $\xi\eta\zeta$ the displacement, $(fgh) = \dfrac{1}{\mu^2}$ curl $(\xi\eta\zeta)$, c a constant that will prove to be the velocity of light in free ether, whereas μ will be identified with the refractive index.

The dynamical equations and boundary conditions are most simply obtained from the Principle of Action, which makes

$$\delta\int(\mathrm{T}-\mathrm{W})\,dt = 0.$$

We have
$$\delta\int(\mathrm{T}-\mathrm{W})dt = \Sigma\iint\rho\,(\dot\xi\delta\dot\xi - c^2\mu^2 f\delta f)\,dtd\mathrm{V},$$

$$\int\dot\xi\delta\dot\xi dt = \dot\xi\delta\xi - \int\ddot\xi\delta\xi dt.$$

The term containing $\delta\xi$ in the variation of $\delta\mathrm{W}$ is

$$c^2\int\rho\left(g\frac{\partial\delta\xi}{\partial z} - h\frac{\partial\delta\xi}{\partial y}\right)d\mathrm{V} = c^2\int\rho\,(ng-mh)\,\delta\xi d\mathrm{S} - c^2\int\left(\frac{\partial g}{\partial z}-\frac{\partial h}{\partial y}\right)\delta\xi d\mathrm{V},$$

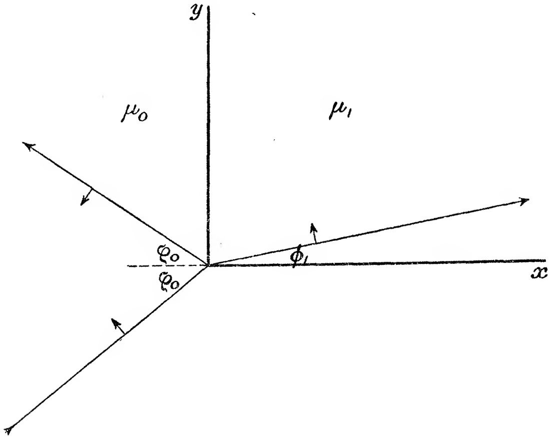
where (l, m, n) are the direction cosines of the outward normal to the bounding surface S.

Picking out the coefficient of $\delta\xi$ in the variation, we get

$$\ddot\xi = c^2\left(\frac{\partial g}{\partial z}-\frac{\partial h}{\partial y}\right), \tag{i}$$

and the surface conditions require $\rho c^2(ng-mh)$, $\rho c^2(lh-nf)$, $\rho c^2(mf-lg)$, to be continuous. The displacement $(\xi\eta\zeta)$ must also be continuous to avoid rupture of the medium. We shall take ρ to be continuous, and if the surface of separation between different media be $x=$constant, the boundary conditions are satisfied when g, h, ξ, η, ζ are continuous. These boundary conditions combined with three dynamical equations of the type (i) will suffice to solve completely the problem of the reflection and refraction of light at the boundary of two transparent isotropic media. We take (f, g, h) to represent the light vector, and if we wish to interpret results in the language of electric theory, we identify (f, g, h) with the electric displacement, and take $\xi\ \dot{\eta}\ \dot{\zeta}$ proportional to the magnetic force.

We shall first consider the case of an abrupt transition from one medium to the other. The surface of separation is $x=0$, the plane of xy is that of incidence, so that everything is independent of z.



(1) Vibrations (of f, g, h) *parallel* to the plane of incidence; so that $(\xi n \zeta)$ is at right angles to this plane.

Thus $\xi = 0 = \eta$;

$$\zeta_0 = e^{ipt-i(x\cos\phi_0+y\sin\phi_0)/V_0} \text{ (incident)}$$

$$+ re^{ipt-i(x\cos\phi'+y\sin\phi')/V_0} \text{ (reflected)};$$

$$\zeta_1 = \frac{\mu_1}{\mu_0} . se^{ipt-i(x\cos\phi_1+y\sin\phi_1)/V_1} \text{ (refracted)}.$$

The boundary conditions require the exponential factor to be the same for all values of t and y when $x=0$; thus

$$\sin\phi_0/V_0 = \sin\phi'/V_0 = \sin\phi_1/V_1,$$

whence $\phi'=180^\circ-\phi_0$, which is the law of reflection; and $\sin\phi_0/V_0=\sin\phi_1/V_1$, which is the law of refraction for $V = c/\mu$ in order that the dynamical equation (i) for S may be satisfied.

$$f = \frac{1}{\mu^2}\frac{\partial S}{\partial y}; \quad g = -\frac{1}{\mu^2}\frac{\partial S}{\partial x}; \quad h = 0.$$

Thus

$$f_0 = -\frac{ip}{c\mu_0}\sin\phi_0\,[e^{ip(\ldots)} + re^{ip(\ldots)}]; \qquad f_1 = -\frac{ip}{c\mu_0}\sin\phi_1\,[se^{ip(\ldots)}];$$

$$g_0 = -\frac{ip}{c\mu_0}\cos\phi_0\,[e^{ip(\ldots)} - re^{ip(\ldots)}]; \qquad g_1 = \frac{ip}{c\mu_0}\cos\phi_1\,[se^{ip(\ldots)}].$$

These represent waves (of f, g, h) whose amplitudes are in the ratios $1:r:s$.

The boundary conditions give $1+r = \mu_1/\mu_0 . s$; $\cos\phi_0\ (1-r) = s\cos\phi_1$; whence, remembering that $\mu_0 \sin\phi_0 = \mu_1 \sin\phi_1$, we get

$$r = \frac{\tan(\phi_0-\phi_1)}{\tan(\phi_0+\phi_1)} \quad \text{and} \quad s = \frac{2\sin\phi_1\cos\phi_0}{\sin(\phi_0+\phi_1)\cos(\phi_0-\phi_1)}.$$

2. Vibrations (of f, g, h) *perpendicular* to the plane of incidence. In this case it is convenient to introduce a new vector (ξ', η', ζ'), of which (ξ, η, ζ) is the curl.

$$\xi' = 0 = \eta'; \quad \xi = \frac{\partial\zeta'}{\partial y}; \quad \eta = -\frac{\partial\zeta'}{\partial x}; \quad \zeta = 0.$$

$$f = 0; \quad g = 0; \quad h = -\frac{1}{\mu^2}\nabla^2\zeta'.$$

Take

$$\zeta'_0 = e^{ipt-i(x\cos\phi_0+y\sin\phi_0)/V_0} + r'e^{ipt+i(x\cos\phi_0-y\sin\phi_0)/V_0}.$$

$$\zeta'_1 = s'e^{ipt-i(x\cos\phi_1+y\sin\phi_1)/V_1}.$$

Then the amplitude of the incident, reflected, and refracted waves (of f, g, h) are in the ratios $1 : r' : s'$.

$$\xi_0 = -\frac{ip}{c}\mu_0\sin\phi_0\,[e^{ip(\ldots)} + r'e^{ip(\ldots)}]; \qquad \xi_1 = -\frac{ip}{c}\mu_1\sin\phi_1\,[s'e^{ip(\ldots)}];$$

$$\eta_0 = \frac{ip}{c}\mu_0\cos\phi_0\,[e^{ip(\ldots)} - r'e^{ip(\ldots)}]; \qquad \eta_1 = \frac{ip}{c}\mu_1\cos\phi_1\,[s'e^{ip(\ldots)}].$$

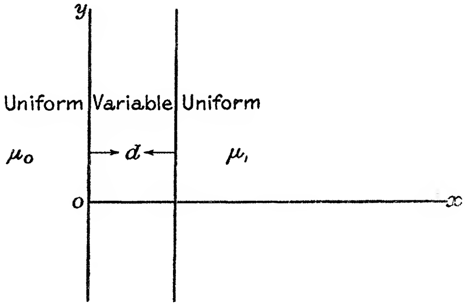
The boundary conditions give $1+r' = s'$; $\mu_0\cos\phi_0(1-r') = \mu_1\cos\phi_1 s'$, whence

$$r' = -\frac{\sin(\phi_0-\phi_1)}{\sin(\phi_0+\phi_1)} \quad \text{and} \quad s' = \frac{2\sin\phi_1\cos\phi_0}{\sin(\phi_0+\phi_1)}.$$

If the incident light is plane polarised at an azimuth of 45° to the plane of incidence, then the amplitudes of the incident light are equal for the vibrations parallel and perpendicular to the plane of incidence. After reflection and refraction the amplitudes will no longer be equal. Let ϵ_1 be the ratio of the amplitude of the reflected light polarised parallel to the plane of incidence to that polarised at right angles, and let ϵ_2 be the same ratio for the refracted light. Then ϵ_1 and ϵ_2 are the "coefficients of ellipticity" of the reflected and refracted light, and we have :—

$$\epsilon_1 = \frac{r}{r'} = -\frac{\cos(\phi_0+\phi_1)}{\cos(\phi_0-\phi_1)} \quad \text{and} \quad \epsilon_2 = \frac{1}{\cos(\phi_0-\phi_1)}.$$

The formulæ for r and r' agree with those of Fresnel. They make ϵ_1 vanish when $\theta_0+\theta_1 = 0$, *i.e.*, when $\theta_0 = \tan^{-1}\mu_1/\mu_0$ the polarising angle; and they indicate a sudden change of phase of 180° (half a wave-length) at this angle. Experiments show that these formulæ represent the facts very well as far as the ellipticity is concerned, except in the neighbourhood of the polarising angle. It is found, however, that ϵ_1 is sensible at all angles, although smallest at the Brewsterian angle, and that the change of phase does not occur suddenly, but enters by degrees. The explanation of this departure from Fresnel's formulæ was long ago ascribed to a gradual rather than an abrupt transition from one medium to the other. This suggestion was strongly supported by Rayleigh's experiment on reflection from water having its surface artificially cleansed, and by the observation made in 1899 by Drude that the ellipticity of the polarisation of the reflected light in the case of a freshly split surface of rock salt is very small, but that it rapidly increases on standing. It seems probable that Fresnel's formulæ are rigorously applicable to the ideal case of an abrupt transition between two isotropic transparent media, and that the departure from these formulæ is due to the fact that in most experiments there is what has been called a "layer of transition" between the media. The mathematical treatment of the problem of the layer has been undertaken by various writers—amongst others Lorenz, Van Ryn, Van Alkemaade, and Drude on the Continent, and in England, in an illustrative way, by Lord Rayleigh, and systematically by G. A. Schott.* Schott's



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* 'Phil. Trans.,' A, 1894, pp. 823 to 885.

analysis is long and complicated, and leads him to results which do not colligate the experimental facts any more closely than very much simpler formulæ that can be obtained. Drude's investigation* is more direct, but from a mathematical point of view his analysis leaves something to be desired, since he treats certain quantities as "approximately constant" merely because they are continuous, and, further, because in proceeding by approximation he gives no indication of the magnitude of the terms neglected.

Take the variable "layer of transition" to extend from $x = 0$ to $x = d$ and to be continuous (as regards μ) with the media bounding. In the layer μ is a function of x only. It will be convenient to put $\mu_0 \sin \phi_0 = \mu_1 \sin \phi_1 = \nu$,

$$\mu_0 \cos \phi_0 = \kappa_0, \qquad \mu_1 \cos \phi_1 = \kappa_1.$$

(1) Vibrations *parallel* to the plane of incidence (*cf.* p. 50).—

In the layer $\zeta = ue^{ip(t-\nu y/c)}$ where u is a function of x only,

$$f = \frac{1}{\mu^2}\frac{\partial \zeta}{\partial y} = \frac{ip\nu}{c\mu^2} ue^{ip(t-\nu y/c)},$$

$$g = -\frac{1}{\mu^2}\frac{\partial \zeta}{\partial x} = -\frac{1}{\mu^2}\frac{du}{dx} e^{ip(t-\nu y/c)}.$$

The equation for S (see (i), p. 49) is $\ddot{\xi} = c^2\left(\frac{\partial f}{\partial y} - \frac{\partial g}{\partial x}\right)$.

Thus $$-p^2 u = c^2\left[-\frac{p^2\nu^2}{c^2\mu^2}u + \frac{d}{dx}\left(\frac{1}{\mu^2}\frac{du}{dx}\right)\right],$$

i.e., $$\frac{d}{dx}\left(\frac{1}{\mu^2}\frac{du}{dx}\right) + \frac{p^2}{c^2}\left(1-\frac{\nu^2}{\mu^2}\right)u = 0.$$

Put $x/d = x_1$ and $d_1 = pd/c = 2\pi d/\lambda_0\mu_0 = 2\pi d/\lambda_1\mu_1$, and we get

$$\frac{d}{dx_1}\left(\frac{1}{\mu^2}\frac{du}{dx_1}\right) + d_1^2\left(1-\frac{\nu^2}{\mu^2}\right)u = 0. \tag{ii}$$

This equation, of course, cannot be solved completely until we know μ^2 as a function of x (and therefore of x_1), *i.e.*, until we know the law of variation of μ in the layer. However, in all cases to which we shall apply the solution, d_1 is a small quantity (as will be shown later), and we can thus solve (ii) by approximations.† We get a first approximation by neglecting d_1^2, so that (ii) becomes

$$\frac{d}{dx_1}\left(\frac{1}{\mu^2}\frac{du}{dx_1}\right) = 0, \qquad \text{whence} \qquad \frac{1}{\mu^2}\frac{du}{dx_1} = \text{B},$$

and $$u = \text{A} + \text{B}\int_0^{x_1} \mu^2 dx_1 = \text{A} + \text{B}\mu^2\text{M},$$

where **M** is a function of x_1.

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* "Lehrbuch der Optik," Transl. Mann and Millikan, p. 288.

† [A somewhat similar analysis has been employed by L. Lorenz.—*Sec.*]

As a next approximation, put $u = \mathrm{A}+\mathrm{B}\mu^2\mathrm{M}+d_1{}^2v$, and the equation for v is

$$\frac{d}{dx}\left(\frac{1}{\mu^2}\frac{dv}{dx_1}\right)+\left(1-\frac{\nu^2}{\mu^2}\right)(\mathrm{A}+\mathrm{B}\mu^2\mathrm{M}) = 0.$$

Integrating, we get

$$\frac{dv}{dx_1}+\mathrm{A}\mu^2x_1+\mathrm{B}\mu^4\mathrm{M}_1-\mathrm{A}\nu^2\mathrm{M}_3-\mathrm{B}\nu^2\mu^2\mathrm{M}_4 = 0,$$

where $\mu^2\mathrm{M}_1 = \int_0^{x_1}\mu^2\mathrm{M}_1dx_1;\qquad \frac{\mathrm{M}_3}{\mu^2}=\int_0^{x_1}\frac{1}{\mu^2}\,dx_1;\qquad \mathrm{M}_4 = \int_0^{x_1}\mathrm{M}dx_1.$

A second integration then gives

$$v+\mathrm{A}\mu^2(x_1\mathrm{M}-\mathrm{M}_1)+\mathrm{B}\mu^4\mathrm{M}_5-\mathrm{A}\nu^2\mathrm{M}_6-\mathrm{B}\nu^2\mu^2\mathrm{M}_7 = 0,$$

where $\mathrm{M}_5 = \int_0^{x_1}\mathrm{M}_1dx_1;\qquad \mathrm{M}_6 = \int_0^{x_1}\mathrm{M}_3dx_1;\qquad \mu^2\mathrm{M}_7 = \int_0^{x_1}\mu^2\mathrm{M}_4dx_1.$

When $x_1 = 0$ all the M's are zero, and when $x_1 = 1$ we have

$\mathrm{M}=\mathrm{E};\quad \mathrm{M}_1 = \mathrm{H};\quad \mathrm{M}_3 = \mathrm{F};\quad \mathrm{M}_4 = \mathrm{J};\quad \mathrm{M}_5 = \mathrm{K};\quad \mathrm{M}_6 = \mathrm{L};\quad \mathrm{M}_7 = \mathrm{N};$

where E . . . N are constants depending on the law of distribution of μ in the layer.

Thus, when $x_1 = 0$ we have

$$u = \mathrm{A};\qquad du/dx_1 = \mathrm{B}\mu_0{}^2;$$

and when $x_1 = 1$,

$$u = \mathrm{A}+\mathrm{B}\mu_1{}^2\mathrm{E}+d_1{}^2\left[\mathrm{A}\{\mu_1{}^2\overline{\mathrm{H}-\mathrm{E}}+\nu^2\mathrm{L}\}+\mathrm{B}\mu_1{}^2\{\nu^2\mathrm{N}-\mu_1{}^2\mathrm{K}\}\right];$$

$$\frac{du}{dx_1} = \mathrm{B}\mu_1{}^2+d_1{}^2\left[\mathrm{A}(\nu^2\mathrm{F}-\mu_1{}^2)+\mathrm{B}\mu_1{}^2(\nu^2\mathrm{J}-\mu_1{}^2\mathrm{H})\right].$$

The boundary conditions give

$$\mathrm{A} = 1+r;\qquad \mathrm{B} = -id_1\kappa_0/\mu_0{}^2\,.\,(1-r);$$

$$\mu_1/\mu_0\,.\,s = \mathrm{A}+\mathrm{B}\mu_1{}^2\mathrm{E}+d_1{}^2\left[\mathrm{A}\{\mu_1{}^2\overline{\mathrm{H}-\mathrm{E}}+\nu^2\mathrm{L}\}+\mathrm{B}\mu_1{}^2\{\nu^2\mathrm{N}-\mu_1{}^2\mathrm{K}\}\right]$$

$$-i\kappa_1\mu_1/\mu_0\,.\,s = \mathrm{B}\mu_1{}^2/d_1+d_1\left[\mathrm{A}(\nu^2\mathrm{F}-\mu_1{}^2)+\mathrm{B}\mu_1{}^2(\nu^2\mathrm{J}-\mu_1{}^2\mathrm{H})\right].$$

Eliminating A and B from these, we get two equations to determine r and s. It will appear later that in most cases we can safely neglect terms containing $d_1{}^2$ and higher powers of d_1. If we do this we get

$$\mu_1/\mu_0\,.\,s-r\left[1+id_1\kappa_0\mathrm{E}\mu_1{}^2/\mu_0{}^2\right] = 1-id_1\kappa_0\mathrm{E}\mu_1{}^2/\mu_0{}^2;$$

$$\kappa_1\mu_1/\mu_0\,.\,s+r\left[\mu_1{}^2/\mu_0{}^2\,.\,\kappa_0-id_1(\nu^2\mathrm{F}-\mu_1{}^2)\right] = \kappa_0\mu_1{}^2/\mu_0{}^2+id_1(\nu^2\mathrm{F}-\mu_1{}^2).$$

Hence

$$s = \mu_0/\mu_1\frac{\begin{array}{c}[1-id_1\kappa_0\mathrm{E}\mu_1{}^2/\mu_0{}^2]\,[\kappa_0\mu_1{}^2/\mu_0{}^2-id_1(\nu^2\mathrm{F}-\mu_1{}^2)]\\ +[1+id_1\kappa_0\mathrm{E}\mu_1{}^2/\mu_0{}^2]\,[\kappa_0\mu_1{}^2/\mu_0{}^2+id_1(\nu^2\mathrm{F}-\mu_1{}^2)]\end{array}}{\kappa_1[1+id_1\kappa_0\,\mathrm{E}\mu_1{}^2/\mu_0{}^2]+\mu_1{}^2/\mu_0{}^2\,\kappa_0-id_1(\nu^2\mathrm{F}-\mu_1{}^2)}$$

$$= 2\kappa_0\mu_1/\mu_0\{(\kappa_1+\kappa_0\mu_1{}^2/\mu_0{}^2)+id_1(\mathrm{E}\kappa_0\kappa_1\mu_1{}^2/\mu_0{}^2+\mu_1{}^2-\nu^2\mathrm{F})\}^{-1},\quad \text{to our order}$$

$$= \mathrm{S}e^{-i\sigma},$$

where $$S = \frac{2\kappa_0\mu_1/\mu_0}{\kappa_1+\kappa_0\mu_1^2/\mu_0^2} = \frac{2\sin\phi_0\cos\phi_0}{\sin(\phi_0+\phi_1)\cos(\phi_0-\phi_1)},$$

and $$\tan\sigma = \frac{E\kappa_0\kappa_1\mu_1^2/\mu_0^2+\mu_1^2-\nu^2F}{\kappa_1+\kappa_0\mu_1^2/\mu_0^2}\,d_1.$$

Thus (to this order) the amplitude is the same as in the ideal case of an abrupt transition; but there is a *small* change of phase. Similarly

$$\begin{aligned} r &= \frac{\kappa_0\mu_1^2/\mu_0^2+id_1(\nu^2F-\mu_1^2)-\kappa_1(1-id_1\kappa_0E\mu_1^2/\mu_0^2)}{\kappa_1(1+id_1\kappa_0E\mu_1^2/\mu_0^2)+\mu_1^2\kappa_0/\mu_0^2-id_1(\nu^2F-\mu_1^2)} \\ &= Re^{-i\rho}, \end{aligned}$$

where $$R = \frac{\kappa_0\mu_1^2/\mu_0^2-\kappa_1}{\kappa_0\mu_1^2/\mu_0^2+\kappa_1} = \frac{\tan(\phi_0-\phi_1)}{\tan(\phi_0+\phi_1)},$$

and $$\tan\rho = \frac{2\kappa_0\mu_1^2/\mu_0^2\,[E\kappa_1^2+F\nu^2-\mu_1^2]}{\kappa_1^2-(\kappa_0\mu_1^2/\mu_0^2)^2}\,d_1,$$

except at the polarising angle, where $\kappa_0\mu_1^2/\mu_0^2-\kappa_1 = 0$, and we have

$$\rho = \pi/2 \quad\text{and}\quad R = \frac{E\kappa_0\kappa_1\mu_1^2/\mu_0^2+F\nu^2-\mu_1^2}{\kappa_1+\kappa_0\mu_1^2/\mu_0^2}\,d_1.$$

Here again, in the general case, the amplitude is (to our order of approximation) the same as that given by Fresnel's formula, but there is a small change of phase depending on d_1. The most marked departure from Fresnel's formula, however, occurs at the polarising angle. In that case R does not vanish as Fresnel gives, but there is a small amount of residual reflected light and the change of phase is $\pi/2$ (a quarter wave-length).

At the polarising angle

$$\kappa_1 = \kappa_0\mu_1^2/\mu_0^2 = \frac{\mu_1^2}{\sqrt{\mu_0^2+\mu_1^2}}.$$

$$\begin{aligned} R &= \frac{d_1}{2\sqrt{\mu_0^2+\mu_1^2}}[E\mu_1^2+F\mu_0^2-(\mu_1^2+\mu_0^2)] \\ &= \frac{d_1}{2\sqrt{\mu_0^2+\mu_1^2}}\int_0^1\frac{(\mu^2-\mu_0^2)(\mu^2-\mu_1^2)}{\mu^2}dx_1]. \end{aligned}$$

(2) We shall consider in a similar way the case of vibrations *perpendicular* to the plane of incidence (*cf.* p. 51).

In the layer $\quad \zeta' = ue^{ip(t-\nu y/c)};\qquad \xi' = \eta' = 0;$

$$\xi = \frac{\partial\zeta'}{\partial y} = -i\frac{p\nu}{c}ue^{ip(\ldots)};\qquad \eta = -\frac{\partial\zeta'}{\partial x} = \frac{du}{dx}e^{ip(\ldots)}\qquad \zeta = 0;$$

$$f = 0;\ g = 0;\quad h = -\frac{1}{\mu^2}\nabla^2\zeta' = \frac{1}{\mu^2}\left[\frac{p^2\nu^2}{c^2}u-\frac{d^2u}{dx^2}\right]e^{ip(\ldots)}.$$

$$\xi = c^2\left(\frac{\partial g}{\partial z}-\frac{\partial h}{\partial y}\right) = c^2\frac{\partial}{\partial y}\left(\frac{1}{\mu^2}\nabla^2\zeta'\right) \quad \therefore \frac{\partial}{\partial y}\left[\ddot{\zeta}'-\frac{c^2}{\mu^2}\nabla^2\zeta'\right] = 0.$$

$$\ddot{\eta} = c^2\left(\frac{\partial h}{\partial x}-\frac{\partial f}{\partial z}\right) = -c^2\frac{\partial}{\partial x}\left(\frac{1}{\mu}\nabla^2\zeta'\right) \quad \therefore \frac{\partial}{\partial x}\left[\ddot{\zeta}'-\frac{c^2}{\mu^2}\nabla^2\zeta'\right] = 0.$$

These equations are satisfied if $\ddot{\zeta}'-\frac{c^2}{\mu^2}\nabla^2\zeta' = 0$. Hence

$$\frac{d^2u}{dx^2}+\frac{p^2}{c^2}(\mu^2-\nu^2)\,u = 0, \quad \text{or} \quad \frac{d^2u}{dx_1^{\,2}}+d_1^{\,2}(\mu^2-\nu^2)\,u = 0 \,\ldots\ldots\ldots \quad \text{(iii)}.$$

Solving (iii) by approximations, we get first $d^2u/dx_1^{\,2} = 0$, so that $u = \mathrm{A}+\mathrm{B}x_1$ and, as a second approximation,

$$u = \mathrm{A}+\mathrm{B}x_1+d_1^{\,2}v, \quad \text{where} \quad d^2v/dx_1^{\,2}+(\mu^2-\nu^2)(\mathrm{A}+\mathrm{B}x_1) = 0 \;\ldots\; \text{(iv)}.$$

Now $$\int x_1\mu^2dx_1 = x_1\mu^2\mathrm{M}-\int\mu^2\mathrm{M}\,dx_1 = \mu^2(x_1\mathrm{M}-\mathrm{M}_1).$$

Hence on integrating (iv), we get

$$\frac{dv}{dx_1} = \nu^2(\mathrm{A}x_1+\tfrac{1}{2}\mathrm{B}x_1^{\,2})-\mu^2(\mathrm{AM}+\mathrm{B}x_1\mathrm{M}-\mathrm{BM}_1)$$

and $$v = \nu^2(\mathrm{A}x_{\frac{1}{2}}^{\,2}+\mathrm{B}x_{\frac{1}{6}}^{\,3})-\mu^2(\mathrm{AM}_1+\mathrm{B}x_1\mathrm{M}_1-2\mathrm{BM}_2),$$

where $$\mu^2\mathrm{M}_2 = \int_0^{x_1}\mu^2\mathrm{M}_1dx_1.$$

When $x_1 = 0$, $\mathrm{M}_2 = 0$, and when $x_1 = 1$, $\mathrm{M}_2 = \mathrm{G}$. Thus we have

when $x_1 = 0$, $\qquad u = \mathrm{A}$; $\quad du/dx_1 = \mathrm{B}$;

and when $x_1 = 1$,

$$u = \mathrm{A}+\mathrm{B}+d_1^{\,2}[\nu^2(\tfrac{1}{2}\mathrm{A}+\tfrac{1}{6}\mathrm{B})-\mu_1^{\,2}(\mathrm{A}+\mathrm{B}\,.\,\mathrm{H}-2\mathrm{BG})];$$

$$du/dx = \mathrm{B}+d_1^{\,2}[\nu^2(\mathrm{A}+\tfrac{1}{2}\mathrm{B})-\mu_1^{\,2}(\mathrm{A}+\mathrm{B}\,.\,\mathrm{E}-\mathrm{BH})].$$

The boundary conditions give

$$\mathrm{A} = 1+r'; \qquad \mathrm{B} = -id_1\kappa_0(1-r');$$

$$s' = \mathrm{A}+\mathrm{B}+d_1^{\,2}[\nu^2(\tfrac{1}{2}\mathrm{A}+\tfrac{1}{6}\mathrm{B})-\mu_1^{\,2}(\mathrm{A}+\mathrm{B}\,.\,\mathrm{H}-2\mathrm{BG})]$$

$$-id_1\kappa_1s' = \mathrm{B}+d_1^{\,2}[\nu^2(\mathrm{A}+\tfrac{1}{2}\mathrm{B})-\mu_1^{\,2}(\mathrm{A}+\mathrm{B}\,.\,\mathrm{E}-\mathrm{BH})].$$

Eliminating A and B from these equations, we get two equations to determine r' and s'. If, as before, we retain only the first power of d_1, we get

$$s'-r'[1+id_1\kappa_0] = 1-id_1\kappa_0;$$

$$\kappa_1s'+r'[\kappa_0+id_1(\mathrm{E}\mu_1^{\,2}-\nu^2)] = \kappa_0-id_1(\mathrm{E}\mu_1^{\,2}-\nu^2).$$

Thus

$$s' = \frac{[1-id_1\kappa_0][\kappa_0+id_1(\mathrm{E}\mu_1^2-\nu^2)]+[1+id_1\kappa_0][\kappa_0-id_1(\mathrm{E}\mu_1^2-\nu^2)]}{[\kappa_0+id_1(\mathrm{E}\mu_1^2-\nu^2)]+\kappa_1[1+id_1\kappa_0]}$$

$$= 2\kappa_0/\{(\kappa_0+\kappa_1)+id_1(\kappa_0\kappa_1+\mathrm{E}\mu_1^2-\nu^2)\}$$

$$= \mathrm{S}'e^{-i\sigma'},$$

where
$$\mathrm{S}' = \frac{2\kappa_0}{\kappa_0+\kappa_1} = \frac{2\sin\phi_0\cos\phi_0}{\sin(\phi_0+\phi_1)},$$

and
$$\tan\sigma' = \frac{\kappa_0\kappa_1+\mathrm{E}\mu_1^2-\nu^2}{\kappa_0+\kappa_1}.$$

Similarly
$$r' = \frac{\kappa_0-id_1(\mathrm{E}\mu_1^2-\nu^2)-\kappa_1(1-id_1\kappa_1)}{\kappa_0+\kappa_1+id_1(\kappa_0\kappa_1+\mathrm{E}\mu_1^2-\nu^2)} = \mathrm{R}'e^{-i\rho'},$$

where
$$\mathrm{R}' = -\frac{\kappa_1-\kappa_0}{\kappa_1+\kappa_0} = -\frac{\sin(\phi_1-\phi_0)}{\sin(\phi_1+\phi_0)}$$

and
$$\tan\rho' = \left[\frac{\kappa_0\kappa_1+\mathrm{F}\mu_1^2-\nu^2}{\kappa_1+\kappa_0}+\frac{\kappa_0\kappa_1-\mathrm{E}\mu_1^2+\nu^2}{\kappa_1-\kappa_0}\right]d_1$$

$$= \frac{2\kappa_0(\kappa_1^2+\nu^2-\mathrm{E}\mu_1^2)}{\kappa_1^2-\kappa_0^2}d_1 = \frac{2\kappa_0\mu_1^2(1-\mathrm{E})}{\mu_1^2-\mu_0^2}d_1.$$

As before, the amplitude is the same as that given by Fresnel, but there is a small change of phase.

If Δ_1 be the difference of phase between the parallel and perpendicular vibrations for reflection, and Δ_2 that for refraction, we have

$$\tan\Delta_1 = \tan(\rho-\rho') = \tan\rho-\tan\rho',\text{ since }\rho\text{ and }\rho'\text{ are small.}$$

$$= 2\kappa_0\mu_1^2d_1\left[\frac{\mathrm{E}\kappa_1^2+\mathrm{F}\nu^2-\mu_1^2}{\mu_0^2\{-(\kappa_0\mu_1^2/\mu_0^2)^2+\kappa_1^2\}}-\frac{1-\mathrm{E}}{\mu_1^2-\mu_0^2}\right],$$

$$\tan\Delta_2 = \tan(\sigma-\sigma') = \tan\sigma-\tan\sigma'.$$

$$= d_1\left[\frac{\mathrm{E}\kappa_0\kappa_1\mu_1^2/\mu_0^2+\mu_1^2-\nu^2\mathrm{F}}{\kappa_1+\kappa_0\mu_1^2/\mu_0^2}-\frac{\kappa_0\kappa_1+\mathrm{E}\mu_1^2-\nu^2}{\kappa_0+\kappa_1}\right],$$

except near the polarising angle, where $\rho = \pi/2$ and $\Delta_1 = \pi/2-\rho'$.

The above results show that ϵ_1 and ϵ_2 are the same as given by Fresnel's formulæ, except in the neighbourhood of the polarising angle when ϵ_1 is not zero, but is very small. At the polarising angle we have

$$\epsilon_1 = \frac{\mathrm{R}}{\mathrm{R}'} = -\frac{d_1}{2}\frac{\sqrt{\mu_1^2+\mu_0^2}}{\mu_1^2-\mu_0^2}\int_0^1\left[\mu^2+\frac{\mu_1^2\mu_0^2}{\mu^2}-(\mu_0^2+\mu_1^2)\right]dx_1.$$

We see from this that ϵ_1 is a maximum along with $\mu^2+\frac{\mu_1^2\mu_0^2}{\mu^2}$, *i.e.*, when $\mu^2 - \mu_0\mu_1$. If μ^2 had this value throughout, we should have

$$\epsilon_1 = \frac{d_1}{2}\frac{\mu_1-\mu_0}{\mu_1+\mu_0}\sqrt{\mu_1^2+\mu_0^2} = \frac{\pi d}{\lambda_0\mu_0}\frac{\mu_1-\mu_0}{\mu_1+\mu_0}\sqrt{\mu_1^2+\mu_0^2}.$$

This enables us to obtain an upper limit for the value of d/λ_0 necessary to produce the observed ellipticity at the polarising angle. Taking Kurz's experiments on reflection from glass into air, we have $\mu_0 = 1$, $\mu_1 = 1{\cdot}5963$, $\epsilon_1 = \tan 1^\circ\ 58' = 0{\cdot}0344$. This gives $d/\lambda_0 = 0{\cdot}0025$ and $d_1 = 0{\cdot}0159$. With Jamin's experiments on reflection from diamond into air $\mu_0 = 1$, $\mu_1 = 2{\cdot}434$, $\epsilon_1 = \tan 1^\circ\ 21' = 0{\cdot}0236$, which gives $d/\lambda_0 = 0{\cdot}0074$ and $d_1 = 0{\cdot}0464$. In the case of glass d_1^2 is about one-hundredth of ϵ_1, while with diamond d_1^2 is about one-tenth of ϵ_1. In the latter case it would not be satisfactory for accurate work to neglect d_1^2 in the determination of the ellipticity, so that it may be advisable to conduct the calculations to a higher order of approximation, retaining d_1^2.

We have

$$\mu_1/\mu_0\,.\,s - r\,[1 + id_1\kappa_0\mathrm{E}\,\mu_1^2/\mu_0^2 + d_1^2\,(\mu_1^2\overline{\mathrm{H}-\mathrm{E}} + \nu^2\mathrm{L})]$$
$$= 1 - id_1\kappa_0\mathrm{E}\,\mu_1^2/\mu_0^2 + d_1^2\,(\mu_1^2\overline{\mathrm{H}-\mathrm{E}} + \nu^2\mathrm{L})\,;$$

$$\kappa_1\,\mu_1/\mu_0\,.\,s + r\,[\mu_1^2\kappa_0/\mu_0^2 - id_1\,(\nu^2\mathrm{F} - \mu_1^2) + \mu_1^2/\mu_0^2\,.\,\kappa_0 d_1^2\,(\nu^2\mathrm{J} - \mu_1^2\mathrm{H})]$$
$$= \kappa_0\mu_1^2/\mu_0^2 + id_1(\nu^2\mathrm{F} - \mu_1^2) + \mu_1^2/\mu_0^2\,.\,\kappa_0 d_1^2\,(\nu^2\mathrm{J} = \mu_1^2\mathrm{H}).$$

$$\therefore\ s = \frac{\mu_1}{\mu_0}\,\frac{\begin{gathered}[1 - id_1\kappa_0\mathrm{E}\mu_1^2/\mu_0^2 + d_1^2\,(\mu_1^2\overline{\mathrm{H}-\mathrm{E}} + \nu^2\mathrm{L})]\\ [\kappa_0 - id_1\mu_0^2/\mu_1^2\,.\,(\nu^2\mathrm{F} - \mu_1^2) + \kappa_0 d_1^2\,(\nu^2\mathrm{J} - \mu_1^2\mathrm{H})]\\ + [1 + id_1\kappa_0\mathrm{E}\mu_1^2/\mu_0^2 + d_1^2\,(\mu_1^2\overline{\mathrm{H}-\mathrm{E}} + \nu^2\mathrm{L})]\\ [\kappa_0 + id_1\mu_0^2/\mu_1^2\,.\,(\nu^2\mathrm{F} - \mu_1^2) + \kappa_0 d_1^2\,(\nu^2\mathrm{J} - \mu_1^2\mathrm{H})]\end{gathered}}{\begin{gathered}\kappa_0\,[1 + id_1\kappa_0\mathrm{E}\mu_1^2/\mu_0^2 + d_1^2\,(\mu_1^2\overline{\mathrm{H}-\mathrm{E}} + \nu^2\mathrm{L})]\\ + \mu_1^2/\mu_0^2\,.\,[\kappa_0 - id_1\mu_0^2/\mu_1^2\,.\,(\nu^2\mathrm{F} - \mu_1^2) + \kappa_0 d_1^2\,(\nu^2\mathrm{J} - \mu_1^2\mathrm{H})]\end{gathered}}$$

$$= \frac{2\kappa_0\mu_1/\mu_0\,.\,[1 + d_1^2\,(\nu^2\overline{\mathrm{L} + \mathrm{J} + \mathrm{EH}} - 2\mathrm{E}\mu_1^2)]}{\begin{gathered}(\kappa_1 + \kappa_0\mu_1^2/\mu_0^2) + id_1\,(-\nu^2\mathrm{F} + \mu_1^2 + \mathrm{E}\kappa_0\kappa_1\mu_1^2/\mu_0^2)\\ + d_1^2\,[\kappa_1\,(\mu_1^2\overline{\mathrm{H}-\mathrm{E}} + \nu^2\mathrm{L}) + \kappa_0\mu_1^2/\mu_0^2\,.\,(\nu^2\mathrm{J} - \mu_1^2\mathrm{H})]\end{gathered}}$$

$$= \mathrm{S}e^{-i\sigma},$$

where

$$\mathrm{S} = \frac{2\kappa_0\mu_1/\mu_0}{\kappa_1 + \kappa_0\mu_1^2/\mu_0^2}\Bigg[1 + d_1^2\Bigg\{\nu^2\,(\mathrm{L} + \mathrm{J} + \mathrm{EH}) - 2\mathrm{E}\mu_1^2$$
$$- \frac{\begin{gathered}(\mathrm{E}\kappa_0\kappa_1\mu_1^2/\mu_0^2 - \nu^2\mathrm{F} + \mu_1^2) + 2\,(\kappa_1 + \kappa_0\mu_1^2/\mu_0^2)\,\{\kappa_1\,(\mu_1^2\overline{\mathrm{H}-\mathrm{E}} + \nu^2\mathrm{L})\\ + \kappa_0\mu_1^2/\mu_0^2\,(\nu^2\mathrm{J} - \mu_1^2\mathrm{H})\}\end{gathered}}{2\,(\kappa_1 + \kappa_0\mu_1^2/\mu_0^2)^2}\Bigg]\Bigg\},$$

and

$$\tan\sigma = \frac{\mathrm{E}\kappa_0\kappa_1\mu_1^2/\mu_0^2 + \mu_1^2 - \nu^2\mathrm{F}}{\kappa_1 + \kappa_0\mu_1^2/\mu_0^2}\,d_1.$$

Similarly

$$r = \frac{\begin{array}{c}\kappa_0\mu_1^2/\mu_0^2 + id_1(\nu^2\mathrm{F} - \mu_1^2) + \mu_1^2/\mu_0^2 \,.\, \kappa_0 d_1^2(\nu^2\mathrm{J} - \mu_1^2\mathrm{H}) \\ -\kappa_1[1 - id_1\kappa_0\mathrm{E}\mu_1^2/\mu_0^2 + d_1^2(\mu_1^2\overline{\mathrm{H}-\mathrm{E}} + \nu^2\mathrm{L})]\end{array}}{\begin{array}{c}(\kappa_1 + \mu_1^2/\mu_0^2\kappa_0) + id_1(-\nu^2\mathrm{F} + \mu_1^2 + \mathrm{E}\kappa_0\kappa_1\mu_1^2/\mu_0^2) \\ + d_1^2[\kappa_1(\mu_1^2\overline{\mathrm{H}-\mathrm{E}} + \nu^2\mathrm{L}) + \kappa_0\mu_1^2/\mu_0^2 \,.\, (\nu^2\mathrm{J} - \mu_1^2\mathrm{H})]\end{array}}$$

$$= \mathrm{R}e^{-i\rho},$$

where

$$\mathrm{R} = \frac{\kappa_0\mu_1^2/\mu_0^2 - \kappa_1}{\kappa_0\mu_1^2/\mu_0^2 + \kappa_1}\left[1 + \frac{2\kappa_0\kappa_1\mu_1^2/\mu_0^2 \,.\, d_1^2}{(\kappa_0^2\mu_1^4/\mu_0^4 - \kappa_1^2)^2}\right.$$

$$\left.\left\{\begin{array}{c}(\kappa_0\kappa_1\mathrm{E}\mu_1^2/\mu_0^2)^2 + (\mu_1^2 - \nu^2\mathrm{F})^2 - \mathrm{E}(\mu_1^2 - \nu^2\mathrm{F})(\kappa_0^2\mu_1^4/\mu_0^4 + \kappa_1^2) \\ + (\kappa_0^2\mu_1^4/\mu_0^4 - \kappa_1^2)(\nu^2 \,.\, \overline{\mathrm{J} - \mathrm{L} + \mu_1^2 \,.\, \mathrm{E} - 2\mathrm{H}})\end{array}\right\}\right]$$

and

$$\tan\rho = \frac{2\kappa_0\mu_1^2/\mu_0^2 \,.\, (\mathrm{E}\kappa_1^2 + \mathrm{F}\nu^2 - \mu_1^2)}{\kappa_1^2 - \kappa_0^2\mu_1^4/u_0^4}\, d_1,$$

except at the polarising angle, where

$$\kappa_1 = \mu_1^2/\mu_0\kappa_0 \quad \text{and} \quad \mathrm{R} = \frac{\kappa_0\kappa_1\mathrm{E}\mu_1^2/\mu_0^2 - \mu_1^2 + \nu^2\mathrm{F}}{\kappa\ + \kappa_0\mu_1^2/\mu_0^2}\, d_1.$$

$$\rho = \frac{\pi}{2} - \theta,$$

$$\tan\theta = d_1\frac{\begin{array}{c}[\kappa_0\mu_1^2/\mu_0^2 \,.\, (\nu^2\mathrm{J} - \mu_1^2\mathrm{H}) - \kappa_1(\mu_1^2\overline{\mathrm{H}-\mathrm{E}} + \nu^2\mathrm{L})](\kappa_1 + \kappa_0\mu_1^2/\mu_0^2) \\ + [(\kappa_0\kappa_1\mathrm{E}\mu_1^2/\mu_0^2)^2 - (\mu_1^2 - \nu^2\mathrm{F})^2]\end{array}}{(\kappa_1 + \kappa_0\mu_1^2/\mu_0^2)(\kappa_0\kappa_1\mathrm{E}\mu_1^2/\mu_0^2 - \mu_1^2 + \nu^2\mathrm{F})}$$

$$= \frac{d_1}{\sqrt{\mu_0^2 + \mu_1^2}}\,\frac{\begin{array}{c}2\mu_1^2[\mathrm{J} - 2\mathrm{H} - \mathrm{L}) + \mu_1^2(\mathrm{E} - 2\mathrm{H})] + \mathrm{E}^2\mu_1^4 \\ -(\mu_0^2 + \mu_1^2 - \mu_0^2\mathrm{F})^2\end{array}}{\mathrm{E}\mu_1^2 - (\mu_0^2 + \mu_1^2 - \mu_0^2\mathrm{F})}.$$

For vibrations perpendicular to the plane of incidence, we have, in like manner,

$$s' = \frac{\begin{array}{l}[1 - id_1\kappa_0 - d_1^2(\mathrm{H}\mu_1^2 - \nu^2/2)][\kappa_0 + id_1(\mathrm{E}\mu_1^2 - \nu^2) + \kappa_0 d_1^2(\nu^2/2 + \overline{\mathrm{H}-\mathrm{E}}\mu_1^2)] \\ + [1 + id_1\kappa_0 - d_1^2(\mathrm{H}\mu_1^2 - \nu^2/2)] \\ \times[\kappa_0 - id_1(\mathrm{E}\mu_1^2 + \nu^2) + \kappa_0 d_1^2(\nu^2/2 + \overline{\mathrm{H}-\mathrm{E}}\mu_1^2)]\end{array}}{\kappa_0 + id_1(\mathrm{E}\mu_1^2 - \nu^2) + \kappa_0 d_1^2(\nu^2/2 + \overline{\mathrm{H}-\mathrm{E}}\mu_1^2) + \kappa_1[1 + id_1\kappa_0 - d_1^2(\mathrm{H}\mu_1^2 + \nu^2/2)]}$$

$$= \frac{2\kappa_0}{(\kappa_0 + \kappa_1) + id_1(\kappa_0\kappa_1 + \mathrm{E}\mu_1^2 - \nu^2) + d_1^2[\overline{\kappa_0 + \kappa_1}\nu^2/2 - \mu_1^2(\mathrm{E}\kappa_0 + \mathrm{H}\overline{\kappa_1 - \kappa_0})]}$$

$$= \mathrm{S}'e^{-i\sigma'},$$

where

$$\mathrm{S}' = \frac{2\kappa_0}{\kappa_0 + \kappa_1}\left[1 - \frac{d_1^2}{2(\kappa_0 + \kappa_1)^2}\right.$$

$$\left.\left\{(\kappa_0\kappa_1 + \mathrm{E}\mu_1^2 - \nu^2)^2 + (\kappa_0 + \kappa_1)(\overline{\kappa_0 + \kappa_1} \,.\, \nu^2 - 2\mu_1^2 \,.\, \overline{\mathrm{E}\kappa_0 + \mathrm{H}\kappa_1 - \mathrm{H}\kappa_0})\right\}\right]$$

$$\tan\sigma' = \frac{\kappa_0\kappa_1 + \mathrm{E}\mu_1{}^2 - \nu^2}{\kappa_0 + \kappa_1}\, d_1,$$

and

$$r' = \frac{\begin{array}{l}[\kappa_0 - id_1(\mathrm{E}\mu_1{}^2 - \nu^2) + \kappa_0 d_1{}^2(\nu^2/2 + \overline{\mathrm{H} - \mathrm{E}\mu_1{}^2})] \\ -\kappa_1[1 - id_1\kappa_0 - d_1{}^2(\mathrm{H}\mu_1{}^2 - \nu^2/2)]\end{array}}{\begin{array}{l}(\kappa_0 + \kappa_1) + id_1(\kappa_0\kappa_1 + \mathrm{E}\mu_1{}^2 - \nu^2) \\ + d_1{}^2[(\kappa_0 + \kappa_1)\nu^2/2 - \mu_1{}^2(\mathrm{E}\kappa_0 + \mathrm{H}\,.\,\overline{\kappa_1 - \kappa_0})]\end{array}}$$

$$= -\frac{\begin{array}{l}(\kappa_1 - \kappa_0) - id_1(\kappa_0\kappa_1 - \mathrm{E}\mu_1{}^2 + \nu^2) \\ + d_1{}^2[(\overline{\kappa_1 - \kappa_0})\nu^2/2 - \mu_1{}^2(\mathrm{E}\kappa_0 - \mathrm{H}\,.\,\overline{\kappa_1 - \kappa_0})]\end{array}}{\begin{array}{l}(\kappa_1 + \kappa_0) + id_1(\kappa_0\kappa_1 + \mathrm{E}\mu_1{}^2 - \nu^2) \\ + d_1{}^2[(\kappa_3 + \kappa_1)\nu^2/2 - \mu_1{}^2(\mathrm{E}\kappa_0 + \mathrm{H}\overline{\kappa_1 - \kappa_0})]\end{array}}$$

$$= \mathrm{R}'e^{-i\rho'},$$

where

$$\mathrm{R}' = \frac{\kappa_1 - \kappa_0}{\kappa_1 + \kappa_0}\Bigg[1 + \frac{2\kappa_0\kappa_1 d_1{}^2}{(\kappa_1{}^2 - \kappa_0{}^2)} \Big\{\kappa_0{}^2\kappa_1{}^2 + (\mathrm{E}\mu_1{}^2 - \nu^2)^2 + \mu_1{}^2(\mathrm{E} - 2\mathrm{H})(\kappa_1{}^2 - \kappa_0{}^2) - (\mathrm{E}\mu_1{}^2 - \nu^2)(\kappa_1{}^2 + \kappa_0{}^2)\Big\}\Bigg],$$

and

$$\tan\rho' = d_1\left[\frac{\kappa_0\kappa_1 + \mathrm{E}\mu_1{}^2 - \nu^2}{\kappa_1 + \kappa_0} + \frac{\kappa_0\kappa_1 - \mathrm{E}\mu_1{}^2 + \nu^2}{\kappa_1 - \kappa_0}\right]$$

$$= 2\kappa_0 d_1\,\frac{\kappa_1{}^2 + \nu^2 - \mathrm{E}\mu_1{}^2}{\kappa_1{}^2 - \kappa_0{}^2} = 2\kappa_0 d_4\,\frac{\mu_1{}^2(1 - \mathrm{E})}{\mu_1{}^2 - \mu_0{}^2}.$$

It will be observed that the retardations of phase (ρ, ρ', σ, σ') are the same as to the first approximation, except that at the polarising angle ρ is $\frac{\pi}{2} - \theta$ instead of $\frac{\pi}{2}$.

On examining the various formulæ thus obtained, it becomes apparent that it will not be possible to calculate the constants d_1, E, F, etc., so as to fit in with experimental results with much accuracy. Of the quantities depending on these constants it is only ϵ_1 (the ellipticity) and ρ the retardation of phase that are large enough to be measured with accuracy, and even with these the departure from Fresnel's formulæ is appreciable only within a few degrees of the polarising angle. Hence, instead of applying these results directly, we shall derive from them much simpler formulæ for ϵ_1 and ρ—formulæ which involve only a single arbitrary constant, and which colligate the experimental results well within the limits of errors of observation.*

In the expression for $\tan\rho$ the factor $2(\mathrm{E}\kappa_1{}^2 + \mathrm{F}\nu^2 - \mu_1{}^2)$ is equal to $-\mu_1{}^2[(\mathrm{F} - \mathrm{E})\cos 2\phi_1 + (2 - \mathrm{E} - \mathrm{F})]$. Now $\mathrm{F} - \mathrm{E}$ is not very large, for F cannot be greater than $\mu_1{}^2$, nor E less than $1/\mu_1{}^2$. Thus for a variation of a few degrees in ϕ_0, the expression (F—E) cos $2\phi_1$ is practically constant. Putting $4d_1\,[(\mathrm{F}-\mathrm{E})\cos 2\phi_1 + 2 - \mathrm{E} - \mathrm{F}] = e$, we have these simple formulæ for ρ and ϵ_1—

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* See the graphs on pp. 64–5.

$$\tan\rho = \frac{e\sin^2\phi_0\cos\phi_0}{\sin 2(\phi_0+\phi_1)\sin 2(\phi_0-\phi_1)}. \qquad \text{(A)}$$

$$\epsilon_1 = \tan\omega = \frac{\mathrm{R}}{\mathrm{R}'} = -\frac{\sin(\phi_0+\phi_1)}{\sin(\phi_0-\phi_1)}\,\frac{\sqrt{(\cos\phi_0 - \mu_0/\mu_1\,.\,\cos\phi_1)^2 + (e/8)^2}}{\cos\phi_0 + \mu_0/\mu_1\,.\,\cos\phi_1}. \qquad \text{(B)}$$

If we put $r = x - iy = \mathrm{R}e^{-i\rho}$, then we have very approximately,

$$x = \frac{\tan(\phi_0-\phi_1)}{\tan(\phi_0+\phi_1)} \quad \text{(the } r \text{ of Fresnel).}$$

$$y = \frac{d_1(\mathrm{E}\kappa_0\kappa_1\mu_1^2/\mu_0^2 + \mathrm{F}\nu^2 - \mu_1^2)}{\kappa_1 + \kappa_0\mu_1^2/\mu_0^2} = \frac{e/8}{\cos\phi_0 + \mu_0/\mu_1\,.\,\cos\phi_1}. \qquad \text{(C)}$$

The denominator in y will vary very little throughout a considerable range on each side of the polarising angle, so that throughout this range y will be very nearly constant. Hence, if we draw a graph to represent the amplitude and phase of the reflected light (the vibrations being parallel to the plane of incidence), we get very approximately a straight line parallel to the axis of x, and all the points of departure from Fresnel's formulæ are indicated—as regards both amplitude and phase—by the slight shifting of a straight line from the axis of x to a parallel position.*

We shall apply these formulæ to the case of reflection and refraction where the media are diamond and air, and compare the theoretical results with Jamin's experiments on reflection with these media.

We have $\mu_0 = 1$; $\mu_1 = 2{\cdot}434$; the polarising angle is 67° 40

Taking $e = 0{\cdot}105$, we get the following table for the ellipticity in the neighbourhood of the polarising angle, the theoretical value being calculated from the formula B, above :—

ϕ_0.	Theory. $\omega = tan^{-1}\epsilon_1$.	Experiment. ω.	Difference.	Theory. ϵ_1^2.	Experiment. ϵ_1^2.	Difference.
66° 0′	−3° 3′	−3° 9′	+ 6′	0·00282	0·00303	−0·00021
66 30	−2 21	−2 23	+ 2	0·00168	0·00173	−0·00005
67 0	−1 45	−1 45	0	0·00092	0·00092	0
67 30	−1 25	−1 22	− 3	0·00061	0·00057	+0·00004
68 0	+1 30	+1 30	0	0·00068	0·00068	0
68 30	+2 2	+2 3	− 1	0·00126	0·00128	−0·00002
69 0	+2 34	+2 57	−23	0·00200	0·00265	−0·00065

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* See graph, p. 65.

Outside of this range the ellipticity is given sufficiently accurately by Fresnel's formula, as the following table will show :—

ϕ_0.	Theory (Fresnel). ω.	Experiment. ω.	Difference.	Theory (Fresnel). ϵ_1^2.	Experiment. ϵ_1^2.	Difference.
60°	−11° 36′	−11° 35′	− 1′	0·04213	0·04202	+0·00011
61	−10 12	−10 1	−11	0·03238	0·03120	+0·00118
62	− 8 48	− 8 18	−30	0·02397	0·02128	+0·00209
63	− 7 21	− 7 36	+15	0·01664	0·01780	−0·00116
64	− 5 50	− 5 46	− 4	0·01042	0·01020	+0·00020
65	− 4 17	− 4 20	+ 3	0·00561	0·00574	−0·00013
70	+ 4 0	+ 4 11	−11	0·00489	0·00535	−0·00046
71	+ 5 47	+ 5 53	− 6	0·01026	0·01062	−0·00036
72	+ 7 35	+ 7 44	− 9	0·01772	0·01844	−0·00072
73	+ 9 25	+ 9 46	−21	0·02751	0·02963	−0·00212
74	+11 18	+12 13	−55	0·03992	0·04688	−0·00696
75	+13 13	+13 30	−17	0·05516	0·05762	−0·00246

The following table gives the retardation of phase ρ calculated from the formula (A) of p. 61. The column headed " ρ " gives the retardation expressed as an angle, that headed " ρ/π " gives the retardation in fractions of the half wave-length :—

ϕ_0.	ρ.	ρ/π.	ϕ_0.	ρ.	ρ/π.
60° 0′	7° 17′	0·0406	68° 0′	112° 26′	0·6243
61 0	8 14	0·0457	68 30	135 50	0·7544
62 0	9 28	0·0527	69 0	149 0	0·8277
63 0	11 14	0·0624	70 0	161 26	0·8966
64 0	13 55	0·0772	71 0	167 8	0·9283
65 0	18 30	0·1028	72 0	170 19	0·9458
66 0	27 46	0·1544	73 0	172 21	0·9570
66 30	36 27	0·2031	74 0	173 48	0·9654
67 0	51 49	0·2877	75 0	174 50	0·9709
67 30	78 30	0·4361			

If $Re^{-i\rho} = x - iy$, we have the following values of x and y calculated from the formula (C) of p. 61 :—

ϕ_0.	x.	y.	ϕ_0.	x.	y.
60°	0·1314	0·01681	68°	−0·0071	0·01727
61	0·1162	0·01689	69	−0·0287	0·01728
62	0·1016	0·01695	70	−0·0514	0·01726
63	0·0858	0·01704	71	−0·0754	0·01723
64	0·0690	0·01710	72	−0·1007	0·01718
65	0·0513	0·01716	73	−0·1247	0·01674
66	0·0328	0·01728	74	−0·1525	0·01695
67	0·0136	0·01726			

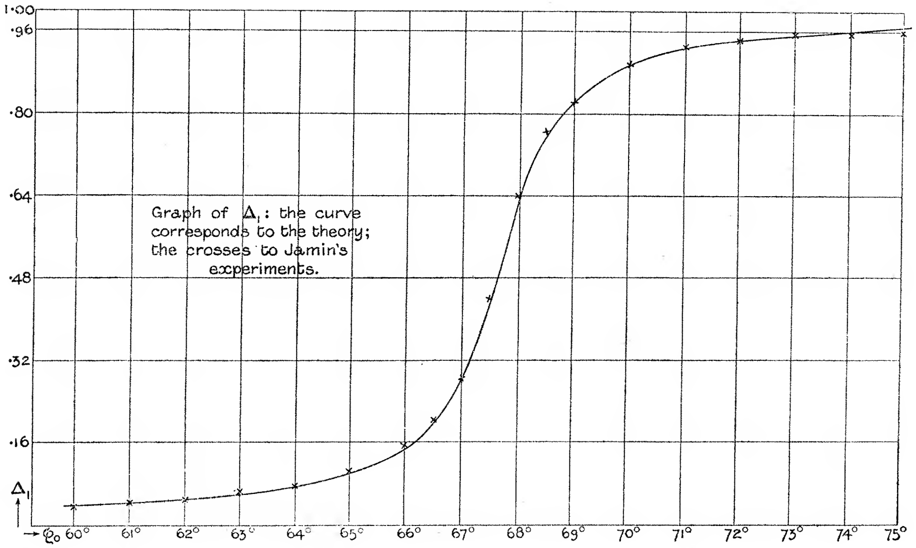
It has been remarked before that the constants d_1, E, F cannot be determined very accurately. We have seen that d_1 is less than 0·046. Further, we have $e/8d_1 = 1-0·8616\text{E}-0·144\text{F}$, and since E and F must be positive, we must have $e/8d_1 < 1$, so that $d_1 > e/8$. If $e = 0·105$, this gives $d_1 > 0·013$, so that d_1 must lie between 0·013 and 0·046.

We shall take $\text{E} = 0·5$, $\text{F} = 1·8$, $d_1 = 0·0412$, and calculate ρ', σ, and σ' from the formulæ of pp. 55 and 57.

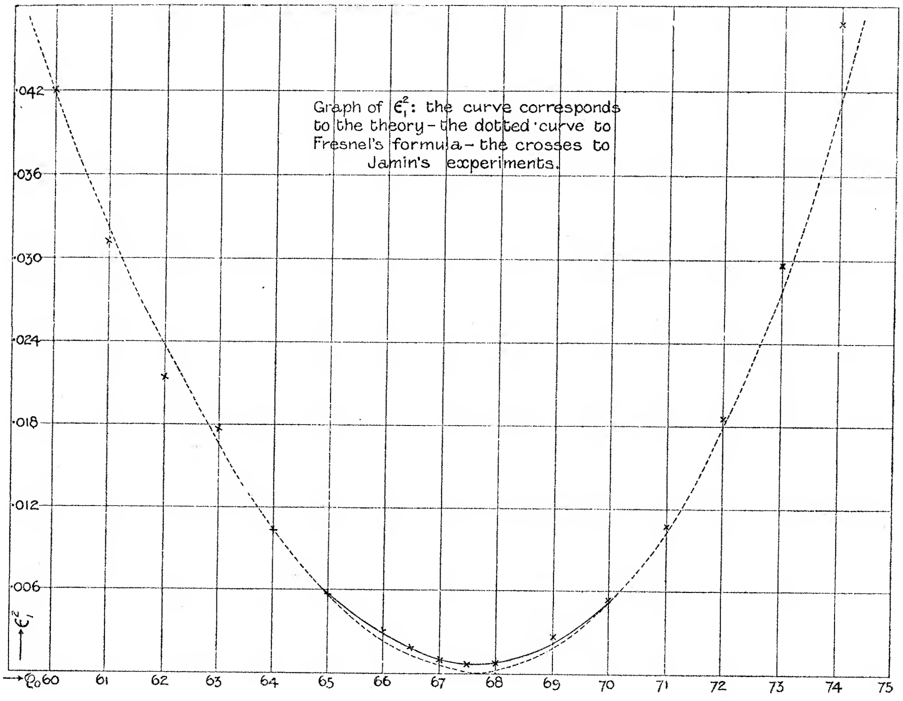
ϕ_0.	$-\sigma$.	σ'.	$-\sigma/\pi$.	σ'/π.	$-\Delta_2 = (\sigma'-\sigma)/\pi$.
60°	1° 12′	6° 0′	0·0067	0·0333	0·0400
61	1 13	6 3	0·0068	0·0336	0·0404
62	1 15	6 5	0·0069	0·0338	0·0407
63	1 16	6 8	0·0070	0·0341	0·0411
64	1 18	6 10	0·0072	0·0342	0·0413
65	1 19	6 13	0·0073	0·0345	0·0418
66	1 21	6 16	0·0075	0·0348	0·0423
67	1 23	6 18	0·0077	0·0350	0·0427
68	1 25	6 21	0·0079	0·0352	0·0431
69	1 26	6 24	0·0080	0·0355	0·0435
70	1 28	6 28	0·0083	0·0359	0·0442
71	1 30	6 29	0·0083	0·0360	0·0443
72	1 32	6 32	0·0085	0·0363	0·0448
73	1 35	6 35	0·0088	0·0366	0·0454
74	1 38	6 38	0·0091	0·0368	0·0459
75	1 40	6 41	0·0092	0·0371	0·0463

The following table gives the value of ρ' and the difference of phase Δ_1 (expressed as a fraction of the half wave-length) between the parallel and perpendicular vibrations, compared with Jamin's observations on reflection:—

ϕ_0.	ρ'.	ρ'/π.	$\Delta_1 = (\rho-\rho')/\pi$. Theory.	Δ_1. Experiment.	Difference.
60° 0′	1° 25′	0·0078	0·0328	0·032	+0·0008
61 0	1 23	0·0076	0·0381	0·042	−0·0039
62 0	1 20	0·0074	0·0453	0·047	−0·0047
63 0	1 17	0·0071	0·0553	0·063	−0·0077
64 0	1 14	0·0069	0·0703	0·073	−0·0027
65 0	1 12	0·0067	0·0961	0·105	−0·0089
66 0	1 9	0·0064	0·1480	0·155	−0·0070
66 30	1 8	0·0063	0·1968	0·202	−0·0052
67 0	1 7	0·0061	0·2816	0·288	−0·0064
67 30	1 5	0·0060	0·4301	0·437	−0·0069
68 0	1 4	0·0059	0·6184	0·640	−0·0216
68 30	1 3	0·0058	0·7486	0·769	−0·0204
69 0	1 1	0·0056	0·8221	0·826	−0·0039
70 0	0 59	0·0055	0·8911	0·897	−0·0069
71 0	0 56	0·0052	0·9231	0·928	−0·0049
72 0	0 53	0·0050	0·9408	0·940	+0·0008
73 0	0 50	0·0046	0·9524	0·948	+0·0044
74 0	0 47	0·0045	0·9609	0·955	+0·0059
75 0	0 44	0·0041	0·9668	0·962	+0·0048



Graph of Δ_1: the curve corresponds to the theory; the crosses to Jamin's experiments.



Graph of ϵ_1^2: the curve corresponds to the theory – the dotted curve to Fresnel's formula – the crosses to Jamin's experiments.

Graph representing the amplitude and phase of the reflected light, the vibrations being parallel to the plane of incidence. OP represents the ratio of the amplitude of the reflected wave to that of the incident wave, and the angle P0x represents the retardation of phase.